3.1 Historical and Mathematical Background of the Wave Principle

Statue of Leonardo Fibonacci, Pisa, Italy.
The inscription reads, "A. Leonardo Fibonacci, Insigne Matematico Piisano del Secolo XII."
Photo by Robert R. Prechter, Sr.

The Fibonacci (pronounced fib-eh-nah´-chee) sequence of numbers was discovered (actually rediscovered) by Leonardo Fibonacci da Pisa, a thirteenth century mathematician. We will outline the historical background of this amazing man and then discuss more fully the sequence (technically it is a sequence and not a series) of numbers that bears his name. When Elliott wrote *Nature's Law*, he explained that the Fibonacci sequence provides the mathematical basis of the Wave Principle. (For a further discussion of the mathematics behind the Wave Principle, see "Mathematical Basis of Wave Theory," by Walter E. White, in a forthcoming book from New Classics Library.)

Leonardo Fibonacci da Pisa

The Dark Ages were a period of almost total cultural eclipse in Europe. They lasted from the fall of Rome in 476 A.D. until around 1000 A.D. During this period, mathematics and philosophy waned in Europe but flowered in India and Arabia since the Dark Ages did not extend to the East. As Europe gradually began to emerge from its stagnant state, the Mediterranean Sea developed into a river of culture that directed the flow of commerce, mathematics and new ideas from India and Arabia.

During the Middle Ages, Pisa became a strongly walled city-state and a flourishing commercial center whose waterfront reflected the Commercial Revolution of that day. Leather, furs, cotton, wool, iron, copper, tin and spices were traded within the walls of Pisa, with gold serving as an important currency. The port was filled with ships ranging up to four hundred tons and eighty feet in length. The Pisan economy supported leather and shipbuilding industries and an iron works. Pisan politics were well constructed even according to today's standards. The Chief Magistrate of the Republic, for instance, was not paid for his services until after his term of office had expired, at which time his administration could be investigated to determine if he had earned his salary. In fact, our man Fibonacci was one of the examiners.

Born between 1170 and 1180, Leonardo Fibonacci, the son of a prominent merchant and city official, probably lived in one of Pisa's many towers. A tower served as a workshop, fortress and family residence and was constructed so that arrows could be shot from the narrow windows and boiling tar poured on strangers who approached with aggressive intent. During Fibonacci's lifetime, the bell tower known as the Leaning Tower of Pisa was under construction. It was the last of the three great edifices to be built in Pisa, as the cathedral and the baptistery had been completed some years earlier.

As a schoolboy, Leonardo became familiar with customs houses and commercial practices of the day, including the operation of the abacus, which was widely used in Europe as a calculator for business purposes. Although his native tongue was Italian, he learned several other languages, including French, Greek and even Latin, in which he was fluent.

Soon after Leonardo's father was appointed a customs official at Bogia in North Africa, he instructed Leonardo to join him in order to complete his education. Leonardo began making many business trips around the Mediterranean. After one of his trips to Egypt, he published his famous *Liber Abaci* (Book of Calculation) which introduced to Europe one of the greatest mathematical discoveries of all time, namely the decimal system, including the positioning of zero as the first digit in the notation of the number scale. This system, which included the familiar symbols 0, 1, 2, 3, 4, 5, 6, 7, 8 and 9, became known as the Hindu-Arabic system, which is now universally used.

Under a true digital or place-value system, the actual value represented by any symbol placed in a row along with other symbols depends not only on its basic numerical value but also on its position in the row, i.e., 58 has a different value from 85. Though thousands of years earlier the Babylonians and Mayas of Central America separately had developed digital or place-value systems of numeration, their methods were awkward in other respects. For this reason, the Babylonian system, which was the first to use zero and place values, was never carried forward into the mathematical systems of Greece, or even Rome, whose numeration comprised the seven symbols I, V, X, L, C, D, and M, with non-digital values assigned to those symbols. Addition, subtraction, multiplication and division in a system using these non-digital symbols is not an easy task, especially when large numbers are involved. Paradoxically, to overcome this problem, the Romans used the very ancient digital device known as the abacus. Because this instrument is digitally based and contains the zero principle, it functioned as a necessary supplement to the Roman computational system. Throughout the ages, bookkeepers and merchants depended on it to assist them in the mechanics of their tasks. Fibonacci, after expressing the basic principle of the abacus in *Liber Abaci*, started to use his new system during his travels. Through his efforts, the new system, with its easy method of calculation, was eventually transmitted to Europe. Gradually Roman numerals were replaced by the Arabic numeral system. The introduction of the new system to Europe was the first important achievement in the field of mathematics since the fall of Rome over seven hundred years before. Fibonacci not only kept mathematics alive during the Middle Ages, but laid the foundation for great developments in the field of higher mathematics and the related fields of physics, astronomy and engineering.

Although the world later almost lost sight of Fibonacci, he was unquestionably a man of his time. His fame was such that Frederick II, a scientist and scholar in his own right, sought him out by arranging a visit to Pisa. Frederick II was Emperor of the Holy Roman Empire, the King of Sicily and Jerusalem, scion of two of the noblest families in Europe and Sicily, and the most powerful prince of his day. His ideas were those of an absolute monarch, and he surrounded himself with all the pomp of a Roman emperor.

The meeting between Fibonacci and Frederick II took place in 1225 A.D. and was an event of great importance to the town of Pisa. The Emperor rode at the head of a long procession of trumpeters, courtiers, knights, officials and a menagerie of animals. Some of the problems the Emperor placed before the famous mathematician are detailed in *Liber Abaci*. Fibonacci apparently solved the problems posed by the Emperor and forever more was welcome at the king’s court. When Fibonacci revised *Liber Abaci* in 1228 A.D., he dedicated the revised edition to Frederick II.

It is almost an understatement to say that Leonardo Fibonacci was the greatest mathematician of the Middle Ages. In all, he wrote three major mathematical works: the *Liber Abaci*, published in 1202 and revised in 1228, *Practica Geometriae*, published in 1220, and *Liber Quadratorum*. The admiring citizens of Pisa documented in 1240 A.D. that he was "a discreet and learned man," and very recently Joseph Gies, a senior editor of the Encyclopedia Britannica, stated that future scholars will in time "give

Leonard of Pisa his due as one of the world's great intellectual pioneers." His works, after all these years, are only now being translated from Latin into English. For those interested, the book entitled *Leonard of Pisa and the New Mathematics of the Middle Ages*, by Joseph and Frances Gies, is an excellent treatise on the age of Fibonacci and his works.

Although he was the greatest mathematician of medieval times, Fibonacci's only monuments are a statue across the Arno River from the Leaning Tower and two streets that bear his name, one in Pisa and the other in Florence. It seems strange that so few visitors to the 179-foot marble Tower of Pisa have ever heard of Fibonacci or seen his statue. Fibonacci was a contemporary of Bonanna, the architect of the Tower, who started building in 1174 A.D. Both men made contributions to the world, but the one whose influence far exceeds the other's is almost unknown.

3.2 The Fibonacci Sequence

In *Liber Abaci*, a problem is posed that gives rise to the sequence of numbers 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, and so on to infinity, known today as the Fibonacci sequence. The problem is this:

How many pairs of rabbits placed in an enclosed area can be produced in a single year from one pair of rabbits if each pair gives birth to a new pair each month starting with the second month?

In arriving at the solution, we find that each pair, including the first pair, needs a month's time to mature, but once in production, begets a new pair each month. The number of pairs is the same at the beginning of each of the first two months, so the sequence is 1, 1. This first pair finally doubles its number during the second month, so that there are two pairs at the beginning of the third month. Of these, the older pair begets a third pair the following month so that at the beginning of the fourth month, the sequence expands 1, 1, 2, 3. Of these three, the two older pairs reproduce, but not the youngest pair, so the number of rabbit pairs expands to five. The next month, three pairs reproduce so the sequence expands to 1, 1, 2, 3, 5, 8 and so forth. Figure 3-1 shows the Rabbit Family Tree with the family growing with exponential acceleration. Continue the sequence for a few years and the numbers become astronomical. In 100 months, for instance, we would have to contend with 354,224,848,179,261,915,075 pairs of rabbits. The Fibonacci sequence resulting from the rabbit problem has many interesting properties and reflects an almost constant relationship among its components.

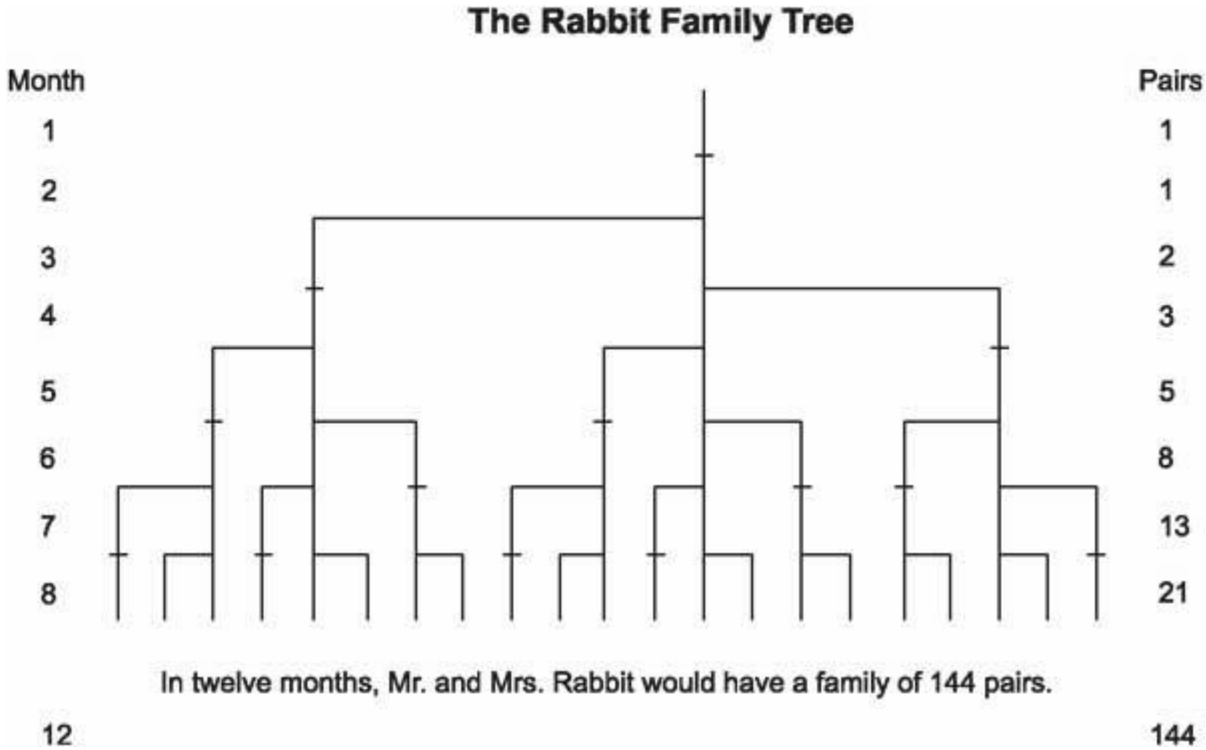

Figure 3-1

The sum of any two adjacent numbers in the sequence forms the next higher number in the sequence, viz., 1 plus 1 equals 2, 1 plus 2 equals 3, 2 plus 3 equals 5, 3 plus 5 equals 8, and so on to infinity.

The Golden Ratio

After the first several numbers in the sequence, the ratio of any number to the next higher is approximately .618 to 1 and to the next lower number approximately 1.618 to 1. The further along the sequence, the closer the ratio approaches *phi*(denoted ϕ) which is an irrational number, .618034.... Between alternate numbers in the sequence, the ratio is approximately .382, whose inverse is 2.618. Refer to Figure 3-2 for a ratio table interlocking all Fibonacci numbers from 1 to 144.

Phi is the only number that when added to 1 yields its inverse: $1 + .618 = 1 \div .618$. This alliance of the additive and the multiplicative produces the following sequence of equations:

$.618^2 = 1 - .618$,
$.618^3 = .618 - .618^2$,
$.618^4 = .618^2 - .618^3$,
$.618^5 = .618^3 - .618^4$, etc.

or alternatively,

$1.618^2 = 1 + 1.618$,
$1.618^3 = 1.618 + 1.618^2$,
$1.618^4 = 1.618^2 + 1.618^3$,
$1.6185^5 = 1.618^3 + 1.618^4$, etc.

Some statements of the interrelated properties of these four main ratios can be listed as follows:

1.618 - .618 = 1,
1.618 x .618 = 1,
1 - .618 = .382,
.618 x .618 = .382,
2.618 - 1.618 = 1,
2.618 x .382 = 1,
2.618 x .618 = 1.618,
1.618 x 1.618 = 2.618.

Besides 1 and 2, any Fibonacci number multiplied by four, when added to a selected Fibonacci number, gives another Fibonacci number, so that:

Fibonacci Ratio Table

DENOMINATOR \ NUMERATOR	1	2	3	5	8	13	21	34	55	89	144
1	1.00	2.00	3.00	5.00	8.00	13.00	21.00	34.00	55.00	89.00	144.00
2	.50	1.00	1.50	2.50	4.00	6.50	10.50	17.00	27.50	44.50	72.00
3	.333	.667	1.00	1.667	2.667	4.33	7.00	11.33	18.33	29.67	48.00
5	.20	.40	.60	1.00	1.60	2.60	4.20	6.80	11.00	17.80	28.80
8	.125	.25	.375	.625	1.00	1.625	2.625	4.25	6.875	11.125	18.00
13	.077	.154	.231	.385	.615	1.00	1.615	2.615	4.23	6.846	11.077
21	.0476	.0952	.1429	.238	.381	.619	1.00	1.619	2.619	4.238	6.857
34	.0294	.0588	.0882	.147	.235	.3824	.6176	1.00	1.618	2.618	4.235
55	.01818	.03636	.0545	.0909	.1455	.236	.3818	.618	1.00	1.618	2.618
89	.011236	.02247	.0337	.05618	.08989	.140	.236	.382	.618	1.00	1.618
144	.006944	.013889	.0208	.0347	.05556	.0903	.1458	.236	.382	.618	1.00

Toward perfect ratios

Figure 3-2

3 x 4 = 12; + 1 = 13,
5 x 4 = 20; + 1 = 21,
8 x 4 = 32; + 2 = 34,
13 x 4 = 52; + 3 = 55,
21 x 4 = 84; + 5 = 89, and so on.

As the new sequence progresses, a third sequence begins in those numbers that are added to the 4x multiple. This relationship is possible because the ratio between *second* alternate Fibonacci numbers is 4.236, where .236 is both its inverse *and* its difference from the number 4. Other multiples produce different sequences, all based on Fibonacci multiples.

We offer a partial list of additional phenomena relating to the Fibonacci sequence as follows:

1) No two consecutive Fibonacci numbers have any common factors.

2) If the terms of the Fibonacci sequence are numbered 1, 2, 3, 4, 5, 6, 7, etc., we find that, except for the fourth Fibonacci number (3), each time a prime Fibonacci number (one divisible only by itself and 1) is reached, the sequence number is prime as well. Similarly, except for the fourth Fibonacci number (3), all composite sequence numbers (those divisible by at least two numbers besides themselves and 1) denote composite Fibonacci numbers, as in the table below. The converses of these phenomena are not always true.

Fibonacci: Prime vs. Composite

P	P	P	X	P		P				P		P			
1	1	2	3	5	8	13	21	34	55	89	144	233	377	610	987
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
			X	C			C	C	C		C		C	C	C

3) The sum of any ten numbers in the sequence is divisible by 11.

4) The sum of all Fibonacci numbers in the sequence up to any point, plus 1, equals the Fibonacci number two steps ahead of the last one added.

5) The sum of the squares of any consecutive sequence of Fibonacci numbers beginning at the first 1 will always equal the last number of the sequence chosen times the next higher number.

6) The square of a Fibonacci number minus the square of the second number below it in the sequence is always a Fibonacci number.

7) The square of any Fibonacci number is equal to the number before it in the sequence multiplied by the number after it in the sequence plus or minus 1. The plus 1 and minus 1 alternate along the sequence.

8) The square of one Fibonacci number F_n plus the square of the next Fibonacci number F_{n+1} equals the Fibonacci number of F_{2n+1}. The formula $F_n^2 + F_{n+1}^2 = F_{2n+1}$ is applicable to right-angle triangles, for which the sum of the squares of the two shorter sides equals the square of the longest side. At right is an example, using F_5, F_6 and $\sqrt{F_{11}}$.

9) One formula illustrating a relationship between the two most ubiquitous irrational numbers in mathematics, *pi* and *phi*, is as follows:

$F_n \approx 100 \times \pi^2 \times \phi^{(15-n)}$, where $\phi = .618...$, n represents the numerical position of the term in the sequence and F_n represents the term itself. In this case, the number "1" is represented only once, so that $F_1 \approx 1$, $F_2 \approx 2$, $F_3 \approx 3$, $F_4 \approx 5$, etc.

For example, let n = 7. Then,

$$F_7 \approx 100 \times 3.1416^2 \times .6180339^{(15-7)}$$
$$\approx 986.97 \times .6180339^8$$
$$\approx 986.97 \times .02129 \approx 21.01 \approx 21$$

10) One mind stretching phenomenon, which to our knowledge has not previously been mentioned, is that the ratios between Fibonacci numbers yield numbers which very nearly are thousandths of other Fibonacci numbers, the difference being a thousandth of a third Fibonacci number, all in sequence (see ratio table, Figure 3-2). Thus, in ascending direction, identical Fibonacci numbers are related by 1.00, or .987 plus .013; adjacent Fibonacci numbers are related by 1.618, or 1.597 plus .021; alternate Fibonacci numbers are related by 2.618, or 2.584 plus .034; and so on. In the descending direction, adjacent Fibonacci numbers are related by .618, or .610 plus .008; alternate Fibonacci numbers are related by .382, or .377 plus .005; second alternates are related by .236, or .233 plus .003; third alternates are related by .146, or .144 plus .002; fourth alternates are related by .090, or .089 plus .001; fifth alternates are related by .056, or .055 plus .001; sixth through twelfth alternates are related by ratios which are themselves thousandths of Fibonacci numbers beginning with .034. It is interesting that by this analysis, the ratio then between thirteenth alternate Fibonacci numbers begins the series back at .001, one thousandth of where it began! On all counts, we truly have a creation of "like from like," of "reproduction in an endless series," revealing the properties of "the most binding of all mathematical relations," as its admirers have characterized it.

Finally, we note that $(\sqrt{5} + 1)/2 = 1.618$ and $(\sqrt{5} - 1)/2 = .618$, where $\sqrt{5} = 2.236$. 5 is the most important number in the Wave Principle, and its square root is a mathematical key to *phi*.

1.618 (or .618) is known as the Golden Ratio or Golden Mean. Its proportions are pleasing to the eye and ear. It appears throughout biology, music, art and architecture. William Hoffer, writing for the December 1975 *Smithsonian Magazine*, said:

...the proportion of .618034 to 1 is the mathematical basis for the shape of playing cards and the Parthenon, sunflowers and snail shells, Greek vases and the spiral galaxies of outer space. The Greeks based much of their art and architecture upon this proportion. They called it "the golden mean."
Fibonacci’s abracadabric rabbits pop up in the most unexpected places. The numbers are unquestionably part of a mystical natural harmony that feels good, looks good and even sounds good. Music, for example, is based on the 8-note octave. On the piano this is represented by 8 white keys, 5 black ones — 13 in all. It is no accident that the musical harmony that seems to give the ear its greatest satisfaction is the major sixth.

The note E vibrates at a ratio of .62500 to the note C.* A mere .006966 away from the exact golden mean, the proportions of the major sixth set off good vibrations in the cochlea of the inner ear — an organ that just happens to be shaped in a logarithmic spiral.
The continual occurrence of Fibonacci numbers and the golden spiral in nature explains precisely why the proportion of .618034 to 1 is so pleasing in art. Man can see the image of life in art that is based on the golden mean.

Nature uses the Golden Ratio in its most intimate building blocks and in its most advanced patterns, in forms as minuscule as microtubules in the brain and the DNA molecule (see Figure 3-9) to those as large as planetary distances and periods. It is involved in such diverse phenomena as quasi crystal arrangements, reflections of light beams on glass, the brain and nervous system, musical arrangement, and the structures of plants and animals. Science is rapidly demonstrating that there is indeed a basic proportional principle of nature. By the way, you are holding this book with two of your *five* appendages, which have *three* jointed parts, five digits at the end, and three jointed sections to each digit, a 5-3-5-3 progression that mightily suggests the Wave Principle.

3.3 The Golden Section

Any length can be divided in such a way that the ratio between the smaller part and the larger part is equivalent to the ratio between the larger part and the whole (see Figure 3-3). That ratio is always .618.

Figure 3-3

The Golden Section occurs throughout nature. In fact, the human body is a tapestry of Golden Sections (see Figure 3-9) in everything from outer dimensions to facial arrangement. "Plato, in his *Timaeus*," says Peter Tompkins, "went so far as to consider phi, and the resulting Golden Section proportion, the most binding of all mathematical relations, and considers it the key to the physics of the cosmos." In the sixteenth century, Johannes Kepler, in writing about the Golden, or "Divine Section," said that it described virtually all of creation and specifically symbolized God’s creation of "like from like." Man is divided at the navel into a Golden Section. The statistical average is approximately .618. The ratio holds true separately for men, and separately for women, a fine symbol of the creation of "like from like." Is mankind’s progress also a creation of "like from like?"